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Quantum revivals in the Jaynes–Cummings model

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Abstract. We obtain an exact integral representation of the sum describing the evolution of the inversion of the two-level atom in the Jaynes-Cummings model when the field is initially prepared in a coherent state or a state generated by classical sources at non-zero temperature. We use the saddle point method to estimate the integrals.

1. Introduction

The Jaynes-Cummings model describes the interaction of a one-mode quantised radiation field with a single two-level atom in the so-called rotating wave approximation. The dynamics of this idealised model is described by the Hamiltonian

$$H = \tfrac{1}{2}\omega_0 S_3 + \tfrac{1}{2}\kappa(S_+ a + S_- a^\dagger) + \omega a^\dagger a. \tag{1.1}$$

The boson operators $a^\dagger$ and a represent creation and annihilation operators of photons, and fermion operators S_3, S_+, S_- act in the space of atom states. Because these operators are coupled through the interaction term, the problem of solving the Heisenberg equation is non-linear. Nevertheless, in a certain situation it is exactly solvable. It can be shown that the operator $J = a^\dagger a + \frac{1}{2}S_3$ commutes with the Hamiltonian and the algebra of operators S_3, $S_+ a$, $S_- a^\dagger$ is closed in the space spanned by the eigenstates of the operator J with the same eigenvalue. Closed solutions can be obtained if the initial state belongs to this space. If initially the electromagnetic field density matrix has only diagonal elements $\rho_n \equiv \rho_{nn}$ and the atom is in the excited state, the evolution of the inversion $\langle S_3(t)\rangle$ and density matrix ρ_n in the Schrödinger picture is given by the equations

$$\langle S_3(t)\rangle = \sum_{n=0}^{\infty} \rho_n(1 - 2\beta_{n+1})$$

$$\rho_n(t) = (1 - \beta_{n+1})\rho_n + \beta_n \rho_{n-1} \tag{1.2}$$

where

$$\beta_n = \frac{\kappa^2 n}{\Delta^2 + \kappa^2 n} \sin^2[\tfrac{1}{2}(\Delta^2 + \kappa^2 n)^{1/2} t].$$

Interest in this model has been recently stimulated by the work of Eberly *et al* (1980), Narozhny *et al* (1980) and Knight and Radmore (1982). They studied the

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solution when the field is prepared in a coherent state. The solution provides evidence of remarkable behaviour in the inversion of the atom. Despite the fact that the field is initially close to classical, the evolution of the inversion has no simple Rabi oscillations. The envelope of these oscillations periodically collapses to zero. Revivals become broader and at longer time overlap each other. For a detailed discussion and graphical representation we refer readers to the work of Yoo *et al* (1981).

We would like to say something about the nature of the revivals. It has been said in many papers (see, e.g., Haroche 1984) that this phenomenon is due to the discreteness of the quantised electromagnetic field in the cavity while the Cummings collapse of the inversion of the atom appears for the stochastic classical field. Let us consider the narrowly distributed Gaussian field with probability of finding 'energy' n equal to

$$P(n)=(\pi 2\sigma)^{1/2}\exp\left(-\frac{(n-\bar{n})^2}{2\sigma}\right). \tag{1.3}$$

If we have a certain physical object which, when it interacts, oscillates with angular frequency n, the average value of its amplitude $\exp(\mathrm{i}nt)$ is obtained by integrating with distribution $P(n)$:

$$\langle \mathrm{e}^{\mathrm{i}nt}\rangle=\mathrm{e}^{\mathrm{i}\bar{n}t}\exp[-(\sigma t^2/2)]. \tag{1.4}$$

The second term in (1.4) describes Cummings collapse of the oscillations. This collapse is due to the dispersion in the field energy. On the contrary, if we have a non-vanishing distribution for only two n we observe an interference pattern which is the simplest form of the revivals. These trivial remarks explain the origins of the behaviour of an atom interacting with the quantised field initially prepared in an n-localised state. The non-linearity of the interaction leads to the non-linear dependence on n of angular frequency $\sqrt{n}$. It involves some additional features like decreasing and broadening of revivals and eventually their complete overlapping for long interaction time. This last feature was studied in detail in the paper of Hioe *et al* (1983).

Rydberg atom experiments in high Q superconducting cavities (Goy *et al* 1983, Moi *et al* 1983, Meschede *et al* 1985) for the first time approach the conditions required to test the Jaynes-Cummings model for a single two-level atom and a single mode of the electromagnetic field (see also a review in Haroche (1984)). The experimental corroboration of the existence of the quantum revivals is extremely difficult. An important requirement is to properly prepare the atom-field system and then let it evolve freely for each incoming atom. The effects of cavity damping should also be considered (Barnett and Knight 1986). The velocity spread of the atoms greater than 1% washes out the phase relationships responsible for the revivals. In Filipowicz *et al* (1986) an experiment is proposed in which the field is built up by the atoms. There is no necessity to prepare the field in the coherent state. However, it needs a high atom flux which technically contradicts their narrow velocity spread.

In § 4 we study analytically a more realistic model of the interaction of an atom with the field generated by the classical sources at non-zero temperature. For the description of such a field see Barnett and Knight (1985). We show that the temperature corresponding to as many as one black-body photon in the cavity significantly decreases Cummings collapse time and broadens the revivals.

The main calculations of the evolution of the atomic inversion reduce to the evaluation of the sum

$$\langle S_3(t)\rangle=\sum_{n=0}^{\infty}\rho_n\cos(\sqrt{n}t). \tag{1.5}$$

An analytic closed-form approximation to the sum with Poissonian distribution $\rho_n = e^{-\bar{n}}(\bar{n}^n/n!)$ was first obtained by Narozhny *et al* (1980) by changing the sum to an integral and approximating the factorial $n!$ by the Stirling formula. The integral was calculated with the saddle point method. The replacement of the discrete sum by an integral is not correct because the phase information after the first collapse is blurred and no revivals occur. Nevertheless, taking into account all saddle points, the right result is obtained (Yoo and Eberly 1985). These approximations are justified if the mean photon number $\bar{n}$ is large enough. In our work we have made some further progress towards a closed analytic evaluation of (1.5). In § 2 we obtain an exact integral representation of the sum (1.5). The saddle point method applied in § 3 to obtain approximate results is easier to handle for our integral than in the paper of Yoo *et al* (1981). Additionally we are able to show why their approximation is so accurate.

2. Integral representation

First we observed that the power series

$$P(t,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n \cos(\sqrt{n}\, t) \tag{2.1}$$

obeys a diffusion-type equation

$$\frac{\partial^2}{\partial t^2} P(t,\lambda) = -\lambda \frac{\partial}{\partial \lambda} P(t,\lambda) \tag{2.2}$$

with boundary conditions

$$P(0,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n = \rho(\lambda)$$

$$P_t(0,\lambda) = 0$$

$$P(t,0) = \rho(0).$$

In order to solve equation (2.2) we perform a Laplace transformation of $P(t,\lambda)$ in t:

$$\tilde{P}(z,\lambda) = \int_0^{\infty} e^{-zt} P(t,\lambda)\, dt. \tag{2.3}$$

The transform $\tilde{P}$ satisfies an equation

$$z^2 \tilde{P}(z,\lambda) - z\rho(\lambda) = -\lambda \frac{\partial}{\partial \lambda} \tilde{P}(z,\lambda) \tag{2.4}$$

with boundary condition

$$\tilde{P}(z,0) = \frac{\rho(0)}{z}. \tag{2.5}$$

It can easily be integrated. The result is

$$\tilde{P}(z,\lambda) = \int_0^{\lambda} \exp[-z^2 \ln(\lambda/x)]\rho(x)\, dx/x. \tag{2.6}$$

For convenience we change the variable in (2.6) $\ln(\lambda/x)=\eta$ and obtain

$$\tilde{P}(z,\lambda)=z\int_0^\infty \exp(-z^2\eta)\rho(\lambda\,\mathrm{e}^{-\eta})\,\mathrm{d}\eta. \tag{2.7}$$

Function (2.7) exists only if $\mathrm{Re}(z^2)>0$ and is not well suited to perform the inverse Laplace transformation. Nevertheless it can easily be continued analytically to an appropriate form. We introduce the function

$$\tilde{P}_\varepsilon(z,\lambda)=-\mathrm{i}z\int_0^\infty \exp[(\mathrm{i}+\varepsilon)z^2\eta]\rho(\lambda\exp[\eta(\mathrm{i}-\varepsilon)])\,\mathrm{d}\eta \tag{2.8}$$

which has poles with positive real part to the right and poles with negative real part to the left of the imaginary axis on the complex z plane. ε is an infinitesimal parameter. This function exists on the imaginary axis so we can perform the inverse transformation. The integration along this axis leaves, on the LHS, only poles which have positive real parts. Thus the inverse Laplace transform of $\tilde{P}_\varepsilon(z,m)$

$$P_\varepsilon(t,\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\tilde{P}_\varepsilon(z,\lambda)\,\mathrm{e}^{zt}\,\mathrm{d}z \tag{2.9}$$

is not the function $P(t,\lambda)$ in the limit $\varepsilon\to 0$. It can easily be shown that

$$\lim_{\varepsilon\to 0}P_\varepsilon(t,\lambda)=\tfrac{1}{2}E(t,\lambda)=\tfrac{1}{2}\sum_{n=0}^{\infty}\rho_n\lambda^n\exp(\mathrm{i}t\sqrt{n}). \tag{2.10}$$

After integration in (2.9) we obtain

$$\begin{aligned}P_\varepsilon(t,\lambda)&=\frac{1}{\sqrt{\pi\mathrm{i}}}\int_0^\infty \exp[-t^2(\varepsilon-\mathrm{i})/4\eta]\rho(\lambda\exp[-\eta(\varepsilon-\mathrm{i})])\frac{t}{4\sqrt{\eta}^3}\,\mathrm{d}\eta\\&=\frac{1}{\sqrt{\pi\mathrm{i}}}\int_0^\infty \exp[(\mathrm{i}-\varepsilon)y^2]\rho(\lambda\exp[(\mathrm{i}-\varepsilon)t^2/4y^2])\,\mathrm{d}y.\end{aligned} \tag{2.11}$$

Our final result

$$\begin{aligned}E(t,\lambda)&=\sum_{n=0}^{\infty}\rho_n\lambda^n\exp(\mathrm{i}t\sqrt{n})\\&=\lim_{\varepsilon\to 0}\frac{2}{\sqrt{\pi\mathrm{i}}}\int_0^\infty \exp[(\mathrm{i}-\varepsilon)y^2]\rho(\lambda\exp[(\mathrm{i}-\varepsilon)t^2/4y^2])\,\mathrm{d}y\end{aligned} \tag{2.12}$$

can be proved directly when one uses the known integral

$$\int_0^\infty \exp(-ay^2-b/y^2)\,\mathrm{d}y=\tfrac{1}{2}(\pi/a)^{1/2}\exp(-2\sqrt{ab})\qquad \mathrm{Re}(a),\mathrm{Re}(b)>0. \tag{2.13}$$

We can rewrite formula (2.12) by

$$E(t,\lambda)=\frac{2}{\sqrt{\pi\mathrm{i}}}\int_P \exp(\mathrm{i}y^2)\rho(\lambda\exp(\mathrm{i}t^2/4y^2))\,\mathrm{d}y \tag{2.14}$$

where P is the path which starts from zero in the fourth quadrant and tends to infinity in the first quadrant of the complex plane of y.

3. The coherent state

In this section we calculate the sum

$$U(t, \bar{n}) = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \exp(\mathrm{i}t\sqrt{n}) \tag{3.1}$$

which corresponds to the expectation value of the inversion of the atom when the field is initially prepared in the coherent state. With the help of formula (2.14) we have the integral representation of (3.1):

$$U(t, \bar{n}) = \frac{2}{\sqrt{\pi \mathrm{i}}} \int_P \exp\{\mathrm{i}y^2 + \bar{n}[\exp(\mathrm{i}t^2/4y^2) - 1]\}\, \mathrm{d}y. \tag{3.2}$$

It is convenient to introduce new scaled variables τ, ξ: $y = \xi(\bar{n}\tau)^{1/2}$, $t = 2\tau(\bar{n})^{1/2}$. The function U depends on τ by the formula

$$U(\tau, \bar{n}) = 2\left(\frac{\bar{n}\tau}{\pi \mathrm{i}}\right)^{1/2} \int_P \exp\{\bar{n}[\mathrm{i}\tau\xi^2 + \exp(\mathrm{i}\tau/\xi^2) - 1]\}\, \mathrm{d}\xi. \tag{3.3}$$

We use the saddle point analysis of the integral representation (3.3). The phase function is given by

$$f(\tau, \xi) = \mathrm{i}\tau\xi^2 + \exp(\mathrm{i}\tau/\xi^2) - 1. \tag{3.4}$$

The saddle points are determined by the condition

$$\frac{\partial}{\partial \xi} f(\tau, \xi)\big|_{\xi = \xi_s} = 0$$

which leads to the following equation

$$\xi_s^4 = \exp(\mathrm{i}\tau/\xi_s^2). \tag{3.5}$$

There is an infinite number of solutions of (3.5) but only those which have a norm close to unity are significant. We can express the function $U(\tau, \bar{n})$ as a sum of contributions from all saddle points:

$$U(\tau, \bar{n}) = \sum_s \frac{\exp(\bar{n} f(x_s))}{(g(x_s))^{1/2}} \tag{3.6}$$

where

$$f(x_s) = \mathrm{i}\tau x_s + x_s^2 - 1$$

$$g(x_s) = 1 + (\mathrm{i}\tau/2x_s)$$

$$x_s^2 = \exp(\mathrm{i}\tau/x_s)$$

$$x_s = \xi_s^2.$$

It can be found that revivals occur when $\tau = \tau_k = 2\pi k$ and that the kth saddle point $x_s = 1$ dominates in the sum (3.6). If we follow Yoo *et al* (1981) and introduce a local time ε_k

$$\varepsilon_k = \tau - 2\pi k \tag{3.7}$$

the function (3.6) can be rewritten

$$U(\tau, \bar{n}) = \sum_{k=0}^{\infty} U_k(\varepsilon_k, \bar{n}) = \sum_{k=0}^{\infty} \frac{\exp[\bar{n} f_k(\varepsilon_k)]}{[g_k(\varepsilon_k)]^{1/2}} \tag{3.8}$$

where

$$f_k(\varepsilon_k) = \mathrm{i}\tau_k + 2\mathrm{i}\varepsilon_k - \frac{1}{2+\mathrm{i}\tau_k}\varepsilon_k^2 + \mathrm{O}(\varepsilon_k^3)$$

$$g_k(\varepsilon_k) = 1 + \mathrm{i}\tfrac{1}{2}\tau_k + \frac{\mathrm{i}}{2+\mathrm{i}\tau_k}\varepsilon_k + \frac{4+\mathrm{i}\tau_k}{2(2+\mathrm{i}\tau_k)^2}\varepsilon_k^2 + \mathrm{O}(\varepsilon_k^3)$$

$$x_k(\varepsilon_k) = 1 + \frac{\mathrm{i}}{2+\mathrm{i}\tau_k}\varepsilon_k + \frac{1}{(2+\mathrm{i}\tau_k)^3}\varepsilon_k^2 + \mathrm{O}(\varepsilon_k^3).$$

In figures 1 and 2 we plot evolution of the inversion together with the accuracy of the method for $\bar{n} = 15$ and evolution of the density matrix ρ for $\bar{n} = 50$.

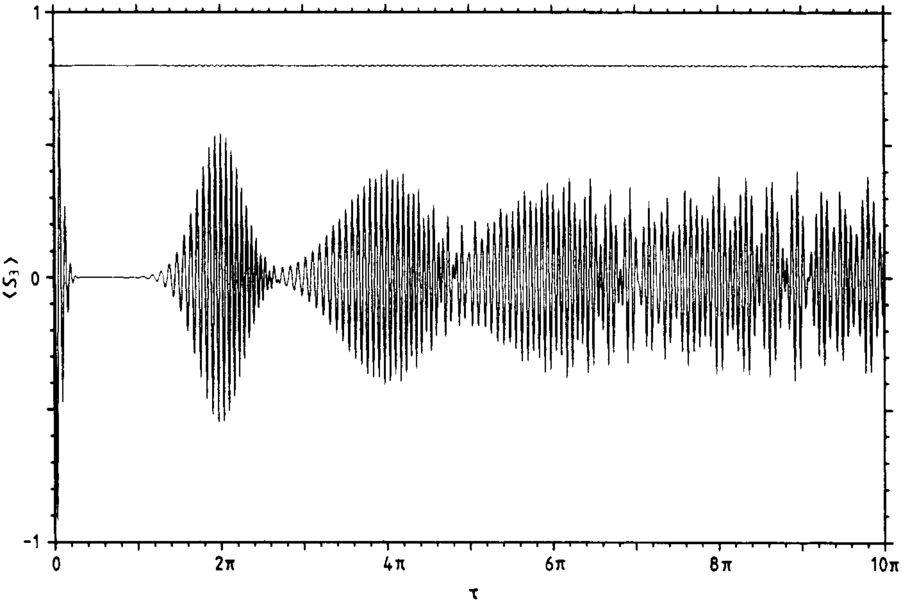

Figure 1. Expectation value of inversion $\langle S_3 \rangle$ as a function of the dimensionless interaction time τ for $\bar{n} = 15$ and $\Delta = 0$.

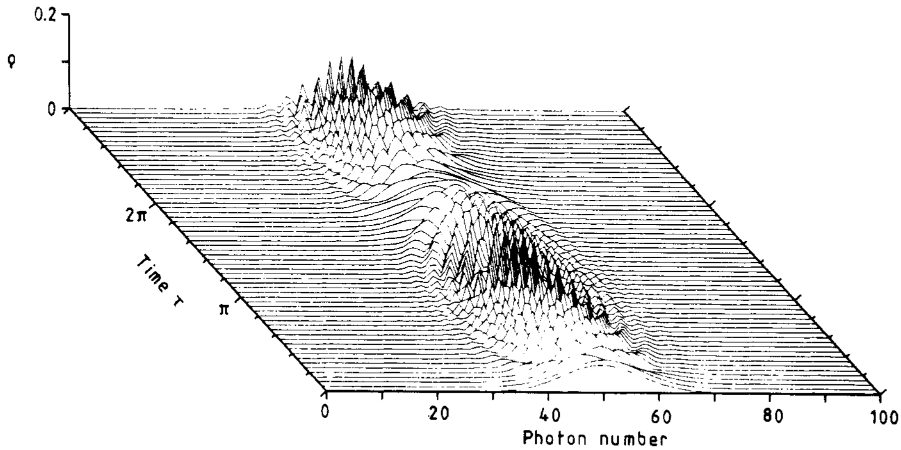

Figure 2. Evolution of the diagonal elements of field density matrix ρ_n for $\bar{n} = 50$ and $\Delta = 0$.

Corresponding results can also be found when the atom is out of resonance. The integral representation of the sum

$$U(t, \bar{n}) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \exp[it(n + \bar{n}\Delta^2)^{1/2}] \tag{3.9}$$

where the detuning Δ is scaled by $\sqrt{\bar{n}}$, is equal to

$$U(\tau, \bar{n}) = 2\left(\frac{\bar{n}\tau}{\pi i}\right)^{1/2} \int_P \exp\{\bar{n}[i\tau\xi^2 + (i\tau/\xi^2)\Delta^2 + \exp(i\tau/\xi^2) - 1]\} \, d\xi. \tag{3.10}$$

The estimate obtained by the saddle point method gives

$$U(\tau, \bar{n}) = \sum_s \frac{\exp[\bar{n} f(x_s)]}{[g(x_s)]^{1/2}} \tag{3.11}$$

where

$$f(x_s) = i\tau x_s(1 + \Delta^2/x_s^2) + x_s^2 - \Delta^2 - 1$$

$$g(x_s) = 1 + \frac{i\tau}{2x_s}\left(1 - \frac{\Delta^2}{x_s^2}\right)$$

$$x_s^2 = \Delta^2 + \exp(i\tau/x_s).$$

The kth saddle point for $\tau = \tau_k = 2\pi k(1 + \Delta^2)^{1/2}$ is equal to $(1 + \Delta^2)^{1/2}$.

It is interesting to see the connection between (3.6) and the equation

$$\frac{\partial^2}{\partial t^2} U(t, \bar{n}) = -\bar{n} U(t, \bar{n}) - \bar{n} \frac{\partial}{\partial \bar{n}} U(t, \bar{n}) \tag{3.12}$$

which is satisfied by the function (3.1). If we seek the solution of (3.12) in the form

$$U(\tau, \bar{n}) = \frac{\exp[\bar{n} f(\tau)]}{\sqrt{g(\tau)}} \tag{3.13}$$

it satisfies equation (3.12) when

$$\bar{n}(\dot{f}^2 + f - \tfrac{1}{2}\tau\dot{f} + 1) + \left(\tfrac{1}{4}\ddot{f} - \frac{\dot{f}\dot{g}}{4g} + \tau\frac{\dot{g}}{4g}\right) + \frac{1}{\bar{n}}\left(-\frac{\ddot{g}}{2g} + \frac{3\dot{g}^2}{4g^2}\right) = 0. \tag{3.14}$$

The functions $f(\tau)$ and $g(\tau)$ described in (3.6) are the solutions of (3.14) when the last term is neglected. This is justified when the mean number of photons $\bar{n}$ is large enough. Nevertheless, (3.13) includes only the first collapse and there is no obvious reason to include all the other revivals.

4. The thermocoherent field

In order to observe revivals one must deal with real final Q resonators and prepare a coherent-type field for each monitoring atom. Here we consider such a resonator driven by an external field, through which a beam of two-level atoms is injected at a flux low enough so that the field can reach the steady state during its damping time. We also assume that the coupling of the atoms to the cavity is sufficiently strong that the damping of the field and driving external field during the interaction can be neglected. Under these assumptions the evolution of the inversion of each atom can be described by JCM with the field being initially prepared in the state which is the steady state solution of the following equation describing the field evolution in the cavity at non-zero temperature driven by external sources:

$$\dot{\rho} = -\mathrm{i}[H_{\mathrm{int}}, \rho] + L\rho \tag{4.1}$$

where the Liouville operator L acting on ρ gives

$$L\rho = \tfrac{1}{2}\gamma(n_{\mathrm{b}}+1)[2a\rho a^{+} - a^{+}a\rho - \rho a^{+}a] + \tfrac{1}{2}\gamma n_{\mathrm{b}}[2a^{+}\rho a - aa^{+}\rho - \rho aa^{+}]$$

and the various symbols have their usual meanings: $H_{\mathrm{int}} = \mathrm{i}\lambda j(a-a^{+}) \equiv \mathrm{i}\gamma\sqrt{\bar{n}}(a-a^{+})$, the interaction Hamiltonian for the external driving field, γ is the damping constant of the cavity, n_{b} is the average black-body photon number in the stationary state without interaction, j is the external current, λ is the coupling constant and $\bar{n} = (\lambda j/\gamma)^2$ is the mean photon number in the stationary state at zero temperature. It is possible to find the steady state solution of (4.1) by transforming the density matrix ρ by a unitary displacement operator U:

$$\rho_U = U\rho U^{-1} \qquad U = \exp[\sqrt{\bar{n}}(a-a^{+})]. \tag{4.2}$$

The transformed density matrix ρ_U satisfies a master equation without the interaction term. The steady state solution is the Boltzmann distribution:

$$\rho_U = \frac{1}{\mathrm{Tr}[\exp(-\beta a^{+}a)]}\exp(-\beta a^{+}a) \tag{4.3}$$

where

$$\mathrm{e}^{-\beta} = n_{\mathrm{b}}/(1+n_{\mathrm{b}}).$$

Finally the steady state solution of equation (4.1) has the following form:

$$\rho = U^{-1}\rho_U U = \frac{1}{\mathrm{Tr}[\exp(-\beta a^{+}a)]}\exp[-\beta(a^{+}-\sqrt{\bar{n}})(a-\sqrt{\bar{n}})]. \tag{4.4}$$

We are interested only in the diagonal elements of ρ in the occupation number representation. After performing some algebra the diagonal elements are equal to

$$\rho_n \equiv \rho_{nn} = \frac{1}{1+n_{\mathrm{b}}}\exp\left(-\frac{\bar{n}}{1+n_{\mathrm{b}}}\right)\left(\frac{n_{\mathrm{b}}}{1+n_{\mathrm{b}}}\right)^{n}\sum_{k=0}^{n}\frac{1}{k!}\binom{n}{k}\left(\frac{\bar{n}}{n_{\mathrm{b}}(1+n_{\mathrm{b}})}\right)^{k}. \tag{4.5}$$

At zero temperature (4.5) tends to a Poissonian distribution with the mean equal to $\bar{n}$:

$$\rho_n = \mathrm{e}^{-\bar{n}}\bar{n}^{n}/n!. \tag{4.6}$$

The calculations of mean photon number and deviation for distribution (4.5) give

$$\begin{aligned}\langle a^{+}a\rangle &= n_{\mathrm{b}} + \bar{n}\\ \sigma &= (\langle (a^{+}a)^2\rangle - \langle a^{+}a\rangle^2)^{1/2} = [\bar{n}(2n_{\mathrm{b}}+1) + n_{\mathrm{b}}^2 + n_{\mathrm{b}}]^{1/2}.\end{aligned} \tag{4.7}$$

From (4.7) it follows that even for $n_{\mathrm{b}} \ll \bar{n}$ the distribution can be much broader than the Poissonian.

We want to calculate the evolution of the inversion when the field is prepared in such a state, which we called the thermocoherent state. Our method developed in § 2 can be applied here without changes. We can find the function $\rho(\lambda)$ for which the Maclaurin expansion is given by coefficients (4.5). The result is

$$\rho(\lambda) = \sum_{k=0}^{\infty}\rho_k\lambda^k = \frac{1}{1+n_{\mathrm{b}}(1-\lambda)}\exp\left[\frac{\bar{n}}{1+n_{\mathrm{b}}}\left(\frac{\lambda}{1+n_{\mathrm{b}}(1-\lambda)}-1\right)\right]. \tag{4.8}$$

Substituting (4.8) into (2.14) we obtain the following integral representation of the sum describing evolution of the inversion of the atom interacting with the thermofield:

$$U(t)=\sum_{n=0}^{\infty}\rho_n\exp(\mathrm{i}t\sqrt{n})$$
$$=\frac{2}{\sqrt{\pi\mathrm{i}}}\int_P \mathrm{d}y\,\exp(\mathrm{i}y^2)\frac{1}{1+n_\mathrm{b}(1-E)}\exp\left[\frac{\bar{n}}{1+n_\mathrm{b}}\left(\frac{E}{1+n_\mathrm{b}(1-E)}-1\right)\right] \tag{4.9}$$

where

$$E=\exp(\mathrm{i}t^2/4y^2).$$

If we repeat all the steps from § 3 we obtain the approximation

$$U(\tau)=\sum_\mathrm{s}\exp\left(\mathrm{i}(\bar{n}+n_\mathrm{b})x_\mathrm{s}\tau-\bar{n}\frac{1-E}{M}\right)$$
$$\times\left\{M\left[1+\frac{\mathrm{i}\tau}{2x_\mathrm{s}M}\left(1+n_\mathrm{b}+\frac{\bar{n}n_\mathrm{b}}{\bar{n}+n_\mathrm{b}}\frac{E^2}{M^2x_\mathrm{s}^2}\right)\right]^{1/2}\right\}^{-1} \tag{4.10}$$

where

$$E=\mathrm{e}(\mathrm{i}\tau/x_\mathrm{s})\qquad M=1+n_\mathrm{b}(1-E)\qquad t=2\tau(\bar{n}+n_\mathrm{b})^{1/2}$$

and the saddle points satisfy the equation

$$(\bar{n}+n_\mathrm{b})x_\mathrm{s}^2=\frac{\bar{n}E}{M^2}+\frac{n_\mathrm{b}E}{M}.$$

Evolution of the inversion together with the accuracy of the applied method for $\bar{n}=15$ and $n_\mathrm{b}=2$ is shown in figure 3. In figure 4 we plot evolution of the photon statistics

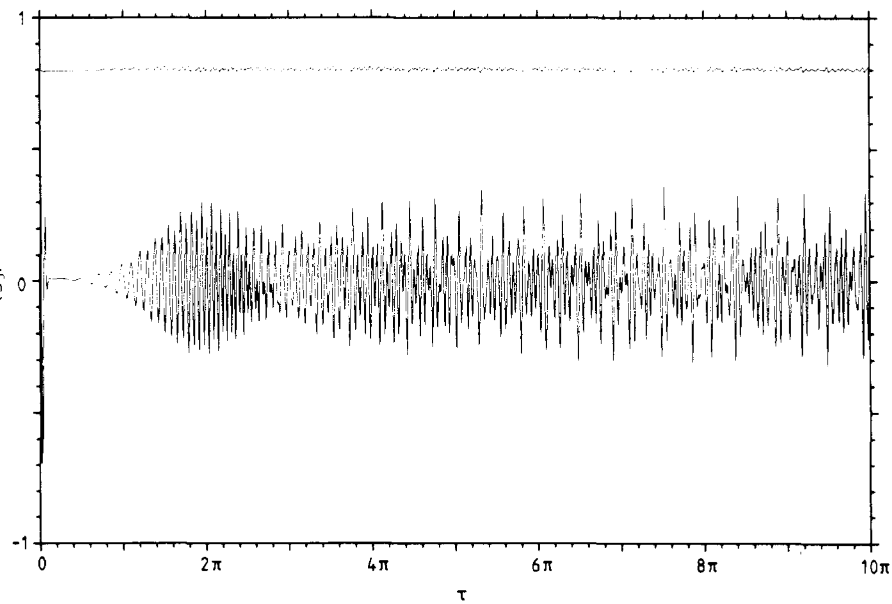

Figure 3. Expectation value of inversion $\langle S_3\rangle$ as a function of the dimensionless interaction time τ for $\bar{n}=15$, $n_\mathrm{b}=2$ and $\Delta=0$.

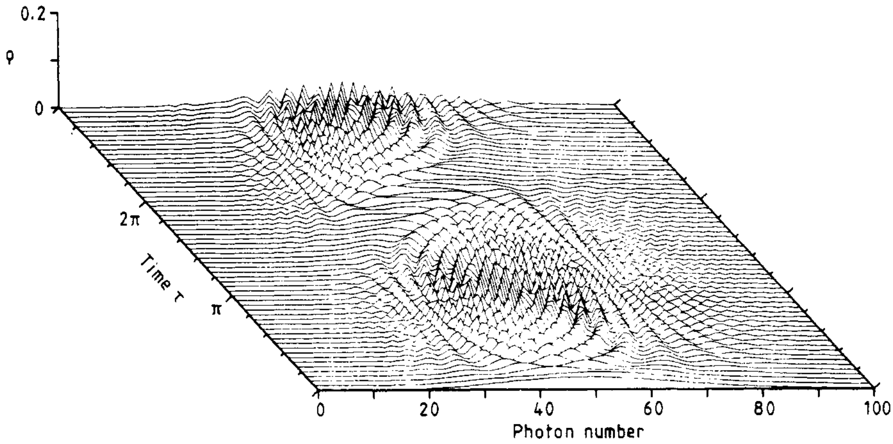

Figure 4. Evolution of the diagonal elements of field density matrix ρ_n for $\bar{n}=50$, $n_b=2$ and $\Delta=0$.

for $\bar{n}=50$ and $n_b=2$. Cummings collapse time is shorter and revivals are less transparent than in the coherent case illustrated in figure 1. The reason is that the photon distribution is much broader.

5. Summary

We have presented the exact integral representation of the power series (2.1). This representation is used to estimate the time dependence of the inversion of the atom when the radiation field is initially prepared in a coherent or a thermocoherent state. Estimations of the integrals are made with the saddle point method. Because the integral representation is valid for all mean photon numbers of the field, we are able to discuss the range of validity of the saddle point method. If higher terms in the expansion of the phase function (3.4) in the saddle point are taken into account, one obtains the expression

$$\begin{aligned} U(\tau, \bar{n}) &= \sum_s \frac{\exp[\bar{n} f(x_s)]}{(g(x_s))^{1/2}} \left[1 + \frac{1}{8\bar{n}} \frac{f^{(4)}(x_s)}{(f^{(2)}(x_s))^2} - \frac{5}{24\bar{n}} \frac{(f^{(3)}(x_s))^2}{(f^{(2)}(x_s))^3} + \mathrm{O}\left(\frac{1}{\bar{n}^2}\right)\right] \\ &= \sum_s \frac{\exp[\bar{n} f(x_s)]}{(g(x_s))^{1/2}} \left[1 + \frac{1}{24\bar{n}} \frac{14x_s + 3\mathrm{i}\tau}{(2x_s + \mathrm{i}\tau)^3} + \mathrm{O}\left(\frac{1}{\bar{n}^2}\right)\right]. \end{aligned} \tag{5.1}$$

As was remarked, only those saddle points which have a norm close to unity are significant in this sum. The correction term in (5.1) is small if $7/96\bar{n} \ll 1$ for $\tau \simeq 0$ and $1/8\bar{n}\tau^2 \ll 1$ for the next revivals. These inequalities explain why the approximations used by Yoo *et al* (1981) are so excellent. They are good enough even if $\bar{n} \simeq 2!$.

For the thermocoherent case a rough estimation of (4.9) shows that for $n_b \ll \bar{n}$ the Cummings collapse time is approximately equal $\tau_c = 1/\{2\pi[\bar{n}(1+2n_b)]^{1/2}\}$ and the kth revival is modulated by the exponential function $\exp\{-2\bar{n}(\tau-\tau_k)^2/[(1+2n_b)\tau_k^2]\}$. From the last expression it is seen that the saddle point method has drawbacks too. First, for large interaction times the revivals become broader and broader so for a given time many saddle points have to be taken into account. Second, if the distribution is so broad that it reaches zero photon number, an infinite number of saddle points should be included independently of the interaction time and this method does not work. In particular it gives wrong results for a pure thermal state.

The method developed here is quite general and can be applied for different distributions $\{\rho_n\}$ if the function $\rho(\lambda)=\Sigma_{k=0}^{\infty}\rho_k\lambda^k$ can be found.

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